

Optimal Classical Random Access Codes Using Single d -level Systems

Andris Ambainis, Dmitry Kravchenko, and Ashutosh Rai

Faculty of Computing, University of Latvia, Raina bulv. 19, Riga, LV-1586, Latvia.

Recently, in the letter [Phys. Rev. Lett. **114**, 170502 (2015)], Tavakoli *et al.* derived interesting results by studying classical and quantum random access codes (RACs) in which the parties communicate higher-dimensional systems. They construct quantum RACs with a bigger advantage over classical RACs compared to previously considered RACs with binary alphabet. However, these results crucially hinge upon an unproven assertion that the classical strategy “majority-encoding-identity-decoding” leads to the maximum average success probability achievable for classical RACs; in this article we provide a proof of this intuition. We characterize all optimal classical RACs and show that indeed “majority-encoding-identity-decoding” is one among the several optimal strategies. Along with strengthening the results in Tavakoli *et al.*, our result provides a firm basis for future research on this topic.

Finding information processing tasks for which quantum effects provide a clear improvement over classical technologies is an underlying theme for a wide range of research problems in quantum information science. These problems, to mention few, cover topics as diverse as designing efficient quantum algorithms [1–4], advancing quantum cryptography [5–8], improving communication complexity using quantum resources [9–11], and numerous other problems in quantum computing and quantum information [12].

This article focuses on random access codes (RACs) [13–25], a communication primitive with a variety of applications in quantum information theory, from the study of quantum automata [13–15] to network coding [18], semi-device independent QKD [20, 21] and foundations of quantum mechanics [22, 23]. We consider the generalized setting for RACs recently introduced and studied by Tavakoli and coauthors [26], which lead them to promising new results on this topic.

In this setting, Alice is given a uniformly random word $x = x_1x_2\dots x_n$ of length n in a d -level alphabet $X = \{1, 2, \dots, d\}$. Bob (who is spatially separated from Alice) receives a uniformly random index $j \in \{1, 2, \dots, n\}$. Bob’s goal is to guess j -th letter x_j of Alice’s word. Alice’s communication to Bob is restricted to sending a single quantum/classical d -level system.

The success in this task is measured by the average probability of correct guess over all possible inputs (x, j) given to Alice and Bob. It is evident that a higher success in this task depends on efficient design of strategies: Alice tries to efficiently encode her input x in a d -level classical/quantum message and send it to Bob, and Bob’s effort is to make the best use of the received message by applying a suitable decoding scheme. A strategy is classical (quantum) optimal if no other classical (quantum) strategy can lead to a higher probability of correct guess.

Knowing the optimal classical strategies is of fundamental importance since it provides the threshold which any meaningful quantum strategy must cross. In what follows, we prove a theorem which firmly establishes the classical optimality of the strategy “majority-encoding-identity-decoding” which is a crucial assumption underlying the strength of quantum RACs developed by Tavakoli

et al. [26]. Moreover, we characterize all optimal classical strategies and show that “majority-encoding-identity-decoding” is one among $(d!)^n$ (or more in few special cases, discussed at the end of this paper) optimal deterministic strategies, where d is the size of the alphabet set X (i.e. $d = |X|$).

Theorem. *Alice sending most frequent letter in her input x and Bob answering with the same letter for any question j is one of optimal deterministic strategies. If $n > 2$ and either $d > 2$ or n is odd, the number of optimal deterministic strategies is exactly $(d!)^n$.*

Proof. An optimal strategy could be either randomized or deterministic. If it is randomized, then there also exists a deterministic optimal strategy. Indeed, the expected success probability of a randomized strategy is nothing but a linear combination of success probabilities of several deterministic strategies. Therefore, at least one of these deterministic strategies must have a success probability that is at least the success probability of the randomized strategy.

We now focus on finding all deterministic optimal strategies. Given a word $x = x_1x_2\dots x_n \in X^n$, Alice sends some $y \in X$; and then Bob answers with $f(y, j) \in X$, where $j \in \{1, 2, \dots, n\}$ is the index provided by the referee. Thus, corresponding to each message y from Alice, Bob must have a string $f(y, 1)f(y, 2)\dots f(y, n)$ and the decoding strategy of Bob is completely described by a $d \times n$ matrix $[f(y, j)]$. Upon receiving a message y from Alice, he looks into the y -th row and answers with the j -th element of that row for the question j . We consider some fragments of such strings separately and denote:

$$\begin{aligned} f_y^j &= f(y, j) \\ f_y^{j\dots k} &= f(y, j)\dots f(y, k) \\ f_y &= f_y^{1\dots n} = f(y, 1)\dots f(y, n) \end{aligned}$$

For a fixed decoding function of Bob, Alice’s best strategy is to send an y_{opt} for which the string $f_{y_{\text{opt}}}$ best approximates word x (of course, Alice can have more than one such optimal letters; in that case she is free to choose any one of them). We now assume that Alice performs the best encoding for any fixed decoding matrix of Bob and

derive the optimal strategies by changing Bob's decoding strategy (decoding matrix).

Definition 1 (Approximation of a string). *We say that a string $z = z_1 z_2 \dots z_n$ d -approximates a word $x = x_1 x_2 \dots x_n$ (or $\text{Sim}(z, x) = d$) if z and x agree in d positions. Formally: $\text{Sim}(z, x) = |\{j : z_j = x_j\}|$.*

Then we can express the value (the success probability) of a strategy f (where Alice applies the best encoding) as follows:

$$\text{Value}(f) = \mathbb{E}_{x \in X^n} \left[\max_y \left(\frac{\text{Sim}(f_y, x)}{n} \right) \right] \quad (1)$$

Since Alice's best choice y is determined by x and $f_t, t \in X$, a strategy f is fully defined by $|X|$ strings $f_t \in X^n$, where t runs through the alphabet X . We now limit our considerations to the case where $|X| \geq 2$ (the game where X consists of only one letter is solved trivially) and where $n > 1$ (the game where $n = 1$ also is solved trivially). Let a and b be arbitrary two letters from the alphabet X .

Lemma 1. *Let f be a strategy. If*

- (i) *there is no y_0 such that $f_{y_0}^1 = a$, and*
- (ii) *there are at least two different $y_1 \neq y_2$ such that $f_{y_1}^1 = f_{y_2}^1 = b$,*

then changing the value of $f_{y_1}^1$ from b to a results in a new strategy g , which is at least as good as f .

Proof.

Definition 2 (Approximation of a word). *We say that a strategy f d -approximates a word x if $\max_{y \in X} \text{Sim}(f_y, x) = d$.*

All words from X^n which do not start from letter a or letter b , are approximated equally by f and g . Indeed: for any such word $x \in X^n$, $\text{Sim}(g_{y_1}, x) = \text{Sim}(a f_{y_1}^{2 \dots n}, x) = \text{Sim}(b f_{y_1}^{2 \dots n}, x) = \text{Sim}(f_{y_1}, x)$.

Some words from X^n which start with letter b might be approximated better by f than by g . E.g., this holds for the word f_{y_1} (unless $f_y = f_{y_1}$ for some $y \neq y_1$). More generally, for any b -word $[?] x$, $\text{Sim}(g_{y_1}, x) = \text{Sim}(f_{y_1}, x) + 1$, since f_{y_1} and g_{y_1} differ only in the first letter.

Assume that x is a b -word which “suffers” from the change $f \rightarrow g$ (that is, $\max_{y \in X} \text{Sim}(g_y, x) = \max_{y \in X} \text{Sim}(f_y, x) - 1$). We claim that the corresponding a -word $x' = a x_2 \dots x_n$ “benefits” from this change (i.e. $\max_{y \in X} \text{Sim}(g_y, x') = \max_{y \in X} \text{Sim}(f_y, x') + 1$).

To see that this is true, we assume the opposite: for the a -word x' , $\max_{y \in X} \text{Sim}(g_y, x') = \max_{y \in X} \text{Sim}(f_y, x') = v$. This can only happen if there exists $y_{\text{opt}} \neq y_1$ such that $\text{Sim}(f_{y_{\text{opt}}}, x') = \text{Sim}(g_{y_{\text{opt}}}, x') = v$ (because for y_1 the two similarities differ by 1). We have

$\text{Sim}(g_{y_{\text{opt}}}, x') \geq \text{Sim}(g_{y_1}, x') > \text{Sim}(f_{y_1}, x')$ (the latter inequality is strict, because x' is an a -word). Hence, $g_{y_{\text{opt}}}$ is good enough approximation also for the b -word x : $\text{Sim}(g_{y_{\text{opt}}}, x') \geq \text{Sim}(f_{y_1}, x') + 1 \implies \text{Sim}(g_{y_{\text{opt}}}, x) \geq \text{Sim}(f_{y_1}, x)$. (By assumption, $g_{y_{\text{opt}}}$ does not start with letter a , so, the similarity with $g_{y_{\text{opt}}}$ does not suffer from replacing the a -word x' by the b -word x . On the other hand similarity with f_{y_1} can benefit at most 1 from that replacement.) But the latter non-strict inequality contradicts the fact that the b -word x has “suffered” from changing f_{y_1} to g_{y_1} (actually, x was not affected at all, because its optimal approximation $f_{y_{\text{opt}}} = g_{y_{\text{opt}}}$ did not change).

To conclude: the change $f \rightarrow g$

- (i) does not affect most words at all, and
- (ii) for each b -word which suffers from that change there is an injectively corresponding a -word which benefits from the change.

Since all words are equiprobable, this means that $\text{Value}(g) \geq \text{Value}(f)$. \square

This proof can be straightforwardly extended to any pair of letters (not only a and b) and to any index j (not only $j = 1$).

Thus for any strategy f we can gradually (without decreasing the value of the strategy at each step) build another deterministic strategy g with the following property:

Property 1 (Condition for the optimality of a strategy). *For each $1 \leq j \leq n$ and $y_1, y_2 \in X$, $y_1 \neq y_2 \implies g_{y_1}^j \neq g_{y_2}^j$.*

Indeed, if there were $y_1 \neq y_2$ such that $g_{y_1}^j = g_{y_2}^j$, then we would have two equal letters $g_{y_1}^j = g_{y_2}^j$ and some missed letter z which is not among g_y^j 's; and we could then make yet another step described in Lemma 1.

We note that the “majority-encoding-identity-decoding” strategy used in Tavakoli *et al.*) is one of strategies that satisfies Property 1. This strategy can be defined by taking $(g_{\text{maj}})_y^j = y$ for each $1 \leq j \leq n, y \in X$ as the decoding function for Bob. Then, the optimal encoding function for Alice is to send the letter y that occurs in the input $x_1 \dots x_n$ most frequently.

As we show next, Property 1 exactly corresponds to the optimality of a strategy (with a small exception: if $|X| = 2$ and n is even, there are some other optimal deterministic strategies).

Lemma 2. *All strategies with Property 1 have the same value.*

Proof. Let g be a strategy with Property 1. For each position j , all letters g_y^j ($y \in X$) are different, so, we have n permutations of letters π_j ($1 \leq j \leq n$): $\pi_j(y) = g_y^j$. Note that there are total $(d!)^n$ number of deterministic strategies which satisfies the Property 1.

We note that the success probability of the strategy g_{maj} if Alice is given $x = x_1 x_2 \dots x_n$ is the same as the success probability of g if Alice is given $x' = g_{x_1}^1 g_{x_2}^2 \dots g_{x_n}^n$. Since the uniform distributions over $x = x_1 x_2 \dots x_n$ and $x' = g_{x_1}^1 g_{x_2}^2 \dots g_{x_n}^n$ are indistinguishable (in both cases, we get a uniformly distributed value from X^n), it follows that the values of g and g_{maj} are equal: $\text{Value}(g) = \text{Value}(g_{\text{maj}})$. \square

Lemma 3. *For any $n > 2$, if $|X| > 2$ or n is odd, then any strategy g with Property 1 will “suffer” from changing value $g_{y_1}^j$ to $g_{y_2}^j$ (assuming $y_1 \neq y_2$).*

Proof. We prove the statement for $y_1 = 1, y_2 = 2$, and $j = 1$, but the proof can be straightforwardly extended to any pair of strings g_{y_1}, g_{y_2} and any index j .

Property 1 means that $g_1^1 \neq g_2^1$. We call these letters a and b , respectively. We now compare the initial strategy g (where $g_1^1 = a, g_2^1 = b$) with the new strategy f (where $f_1^1 = f_2^1 = b$).

We already showed that $\text{Value}(g) \geq \text{Value}(f)$ by describing a mapping from the set of “suffering” b -words to the set of “benefiting” a -words. So now it is sufficient to show that there is no inverse mapping, i.e. that there are more a -“sufferers” than b -“beneficiaries”.

This subset of “suffering” a -words x which does not have their b -“beneficiaries” can be described as follows:

- (i) these a -words x are best approximated by the word g_1 of the strategy g ,
- (ii) the corresponding b -words $bx_2 \dots x_n$ are equidistant from f_1 and f_2 of the strategy f ($\text{Sim}(f_1, x) = \text{Sim}(f_2, x)$), so that the b -word does not “benefit” from the change $g_1 \rightarrow f_1$).

To finish the proof, it suffices to construct just one example of such x . If n is odd, we can take

$$x = a \quad g_1^{2 \dots (n+1)/2} \quad g_2^{(n+3)/2 \dots n}.$$

If n is even, but $|X| > 2$, we round down the fractions and replace the last letter with any letter which is not g_1^n or g_2^n . \square

We just showed that any strategy, which does not satisfy Property 1, can be gradually improved up to a strictly better one with Property 1, and that no better result can be achieved. The two exceptions to this rule are: (i) the case where $n = 2$, and (ii) the case where $|X| = 2$ and n is even. We cover these cases by Lemmata 4 and 5: in both cases optimality is equivalent to Property 1 up to arbitrary change in exactly one column $g_{y_1}^j$ of Bob’s matrix $[g_{y_1}^j]$.

Property 2 (Condition for the optimality of a strategy for (i) $n = 2$ and (ii) for $|X| = 2$ and even n). *For each $1 \leq j \leq n$ (except at most one j_0): for each $y_1, y_2 \in X, y_1 \neq y_2 \implies g_{y_1}^j \neq g_{y_2}^j$.*

Lemma 4. *If $n = 2$, then any strategy g with Property 2 is optimal and is strictly better than any strategy f without Property 2.*

Proof. From Lemmas 1 and 2 it follows that Property 1 implies the optimality of a strategy. Any such strategy 0-approximates exactly d words and 1-approximates all other $d^2 - d$ words from X^2 .

Obviously, any strategy g with Property 2 0-approximates also exactly d words from X^2 , since all g_y ’s are different words (because by Property 2 either all g^1 ’s or all g^2 ’s are different letters).

Obviously, any strategy g with Property 2 also 1-approximates all other $d^2 - d$ words from X^2 . Indeed: either all g^1 ’s or all g^2 ’s are different letters, so for any word $x_1 x_2$ one can choose either such y that $g_y^1 = x_1$ or such y that $g_y^2 = x_2$.

To prove the last part of the statement about a strategy f without Property 2, we note that if there is some missing letter z_1 among f^1 ’s and some missing letter z_2 among f^2 ’s, then word $z_1 z_2$ is only 2-approximated by the strategy f . So we follow that f 0-approximates at most d words from X^2 , and some of remaining $d^2 - d$ words from X^2 cannot be 1-approximated by f . Of course, such strategy f will have strictly lower value than any strategy g with Property 2:

$$\text{Value}(f) < \frac{1 \times d + \frac{1}{2} \times (d-1)d}{d^2} = \text{Value}(g)$$

\square

Lemma 5. *If $|X| = 2$ and $n > 2$ is even, then any strategy g with Property 2 is optimal and is strictly better than any strategy f without Property 2.*

Proof. Note that in this case we have binary alphabet $X = \{0, 1\}$ and only two strings f_0, f_1 of Bob’s strategy f , and this leads to pretty simple combinatorics.

From Lemmas 1 and 2 we obtain that Property 1 implies the optimality of a strategy. Any such strategy g 0-approximates exactly 2 words, 1-approximates exactly $2n$ words, \dots ; generally: k -approximates exactly $2\binom{n}{k}$ words (for $0 \leq k < \frac{n}{2}$). Remaining $\binom{n}{n/2}$ words are k -approximated “twice”: they are equidistant from g_0 and g_1 .

Straightforward consideration shows that any strategy g' with Property 2 also k -approximates exactly $2\binom{n}{k}$ words (for $0 \leq k < \frac{n}{2}$), and remaining $\binom{n}{n/2}$ words are k -approximated (but now only “once”).

Any strategy f without Property 2 k -approximates at most $2\binom{n}{k}$ words (for $0 \leq k < \frac{n}{2}$), and among remaining at least $\binom{n}{n/2}$ words there are some which cannot be $(n/2)$ -approximated. For example, if z_1 is the missing letter among f^1 ’s and z_2 is the missing letter among f^2 ’s, then the word $z_1 z_2$ $f_0^{3 \dots n/2+1} f_1^{n/2+2 \dots n}$ can be only $(n/2 + 1)$ -approximated by the strategy f .

Of course, such strategy f will have strictly lower value than any strategy g with Property 2:

$$\text{Value}(f) < 2^{-n} \sum_{k=0}^n \binom{n}{k} \frac{\max(k, n-k)}{n} = \text{Value}(g)$$

□

To complete the description of the set of all optimal strategies, we should also mention randomized strategies, which consist of an arbitrary set of deterministic strategies with Property 1 with an arbitrary probability distribution on them. □

Concluding remarks—In this article we have characterized all the optimal classical strategies for RACs in which one party receives a random n -length word formed

from d -level alphabets and communicates a single d -level alphabet, and other party tries to guess, at random, the i -th letter in the word. We proved that the strategy “majority-encoding-identity-decoding” is an optimal classical strategy. Regarding evaluating the optimal average success probabilities, we note that a closed analytical formula is hard to derive for general values of parameters n and d . However, it is not hard to find the exact numerical values given the values of parameters n and d in a small range say $1 \leq n, d \leq 100$ and, since quantum RAC protocols are known only for smaller values of n and d , these are the most interesting cases.

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